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2007 J. Phys. A: Math. Theor. 40 14649

(http://iopscience.iop.org/1751-8121/40/49/002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 14649–14665 doi:10.1088/1751-8113/40/49/002

Critical behavior of the heterogeneous random coagulation-fragmentation processes

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Received 2 August 2007, in final form 18 October 2007
Published 21 November 2007
Online at stacks.iop.org/JPhysA/40/14649

Abstract

In this paper, we study the asymptotic distributions of the heterogeneous random coagulation-fragmentation processes (HCFP) which model the coagulation, fragmentation and diffusion of clusters of particles on a lattice. Based on a closed form of the stationary distribution for the HCFP, we prove that the mutually dependent clusters with a finite size (finite particles) in the sub-critical stage will become independent in the critical and super-critical stages, and the asymptotic distributions of the number of clusters may converge to Gaussian or Poisson distribution according to its size, to be small or large in the critical and super-critical stages.

PACS numbers: 82.35.Jk, 02.50.Ga
Mathematics Subject Classification: 60K35

1. Introduction

Since Smoluckowski (1916) proposed the coagulation equations which describe the coupled evolution of the densities $c_j(t)$ of polymers (clusters) made up of j units (particles) ($j = 1, 2, 3, \ldots$) in an infinite-volume homogeneous system, various aspects of the equations and their stochastic counterparts containing the combined effects of coagulation and fragmentation have been extensively studied by many authors (see [2, 3, 6–9, 13–16, 21, 22] and [17]). A detailed overview of the models in the mathematical aspects can be found in [1] or [5]. Recently, a necessary and sufficient condition for the occurrence of a gelation of the reversible random coagulation-fragmentation processes has been studied in [11]. However, the above models have no diffusion of the clusters. If we add the diffusion in the coagulation-fragmentation model such that it has an effective action on the coagulation and fragmentation (i.e. heterogeneous random coagulation-fragmentation processes (HCFP)), can one obtain similar results as the homogeneous coagulation-fragmentation models? Especially, what is the asymptotic behavior of the HCFP? In this paper, we first give a closed form of the stationary distribution for the HCFP, then show that the asymptotical distributions of the number of the clusters may converge to Gaussian or Poisson distribution according to its sizes to be small or large as the total number of particles goes to infinity. Though the process considered in this paper is different from that studied by Pittel *et al* [18, 19], their work provides some good ideas and techniques for us.

Section 2 contains a description of the HCFP. The closed form of the stationary distribution is given in section 3. Our main results about the asymptotical distributions of the number of the clusters are shown in section 4.

2. The heterogeneous coagulation-fragmentation processes

In this section, some notations and preliminaries are given. Consider the following interacting particles system: in each site $x \in \mathbb{Z}^d$ of d-dimensional integer lattice, particles can coagulate to form a cluster, and two clusters can coagulate to form a larger one, and a larger cluster can fragmentate into two smaller ones, and all of them can also move to their neighbor sites y.

Let B be a finite subset of $\mathbb{Z}^d$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$ and $\mathbb{X}(B) = \{A : A \in \mathbb{N}^{B\times\mathbb{N}_+}\}$, where $A = (a(x, k) : a(x, k) \in \mathbb{N}, x \in B, k \in \mathbb{N}_+)$. Here we assume that the number, $|B|$, of set B is at least greater than 2. Thus, A can be regarded as a matrix with indices in $B \times \mathbb{N}_+$. For each $x \in B$, denote by $a(x, k)$ the number of k-clusters in site x. Let $I_{x,i} = (a(y, j) : y \in B, j \in \mathbb{N}_+) \in \mathbb{X}(B)$ be a matrix such that $a(y, j) = 0$ for $(y, j) \neq (x, i)$ and $a(y, j) = 1$ for $(y, j) = (x, i)$. For $A \in \mathbb{X}(B)$, let

$$
\begin{aligned}
&A^k_{x,y} := A + I_{y,k} - I_{x,k}, && \text{if} \quad a(x,k) > 0 \quad \text{and} \quad \|x - y\| = 1;\\
&A^+_{x,ij} := A + I_{x,i+j} - I_{x,i} - I_{x,j}, && \text{if} \quad a(x,i) > 0 \quad \text{and} \quad a(x,j) > 0;\\
&A^-_{x,ij} := A - I_{x,i+j} + I_{x,i} + I_{x,j}, && \text{if} \quad a(x, i+j) > 0;\\
&A_x := (a(x,1), a(x,2), \ldots);
\end{aligned}
$$

$|A_x| = \sum_k ka(x, k)$ and $|A| = \sum_{x\in B} |A_x|$. Here, $A^k_{x,y}$ means that the state of the process obtained from a state A after a jump of a cluster of size k from site x to site y and $A^+_{x,ij}$ denotes that the state obtained from a state A after a cluster of size i coagulate with a cluster of size j to form a cluster of size $i + j$ in the site x, i.e.

$$(i) + (j) \rightarrow (i + j).$$

$A^-_{x,ij}$ means that the state obtained from a state A after a cluster of size $i + j$ breaks into a cluster of size i and a cluster of size j in site x, i.e.

$$(i + j) \rightarrow (i) + (j)$$

and A_x denotes the distribution of the numbers of the different clusters in site x.

Let N denote the total number of particles in the system; then $N = \sum_{x\in B} |A_x|$. Let $\mathbb{X}_N(B) = \{A : A \in \mathbb{X}(B), |A| = N\}$. Now we define the heterogeneous random coagulation-fragmentation process considered in the paper as follows: the process, denoted by $\{A_N(t), t \geqslant 0\}$, is a continuous-time irreducible Markov chain on the finite state space $\mathbb{X}_N(B)$ with state transition rates

$$
Q_{AA'} := \begin{cases}
\frac{1}{2^d} g(a(x,k))/N, & \text{if} \quad A' = A^k_{x,y},\\
K_{ij} g(a(x,i)) g(a(x,j) - \delta_{ij})/N^2, & \text{if} \quad A' = A^+_{x,ij},\\
F_{ij} g(a(x, i+j))/N, & \text{if} \quad A' = A^-_{x,ij},\\
0, & \text{if} \quad A' \neq A^k_{x,y}, A^+_{x,ij}, A^-_{x,ij}
\end{cases}
$$

and

$$Q_{AA} = -\sum_{A' \in \mathbb{X}_N(B), A' \neq A} Q_{AA'},$$

where $A, A' \in \mathbb{X}_N(B)$, $g(.)$ denotes the diffusion rate which is a positive function except $g(0) = 0$, K_{ij} and F_{ij} are coagulation and fragmentation kernels respectively, and $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$. Here the choice of rates of coagulation, fragmentation and diffusion reflect the case like in a polymer system that reaction occurs with a probability proportional to both the numbers of reactants and inversely proportional to the volume; here the density is taken to be equal to 1, so that the volume coincides with the total number of units N. Though there are many ways of action of the diffusion on coagulation and fragmentation, the main reason for us to take the special forms, $g(a(x,i))g(a(x,j) - \delta_{ij})$ and $g(a(x,i+j))$, is that we can easily obtain the stationary distribution of the process by the forms. Note that the coagulation and fragmentation do not depend directly on the diffusion rate when $g(k) = k$.

Remark 1. The coagulation rate will become $\frac{ia(x,i)}{N}\frac{ja(x,j)}{N}$ for $i \neq j$ when $g(k) = k$ and $K_{ij} = ij$ for $i \neq j$ and $K_{ij} = i^2/2$ for $i = j$. In the case of $i \neq j$, $\frac{ia(x,i)}{N}\frac{ja(x,j)}{N}$ denotes the probability of forming $i + j$-cluster from any two i-cluster and j-cluster. For $i = j$, $\frac{1}{2}\frac{ia(x,i)}{N}\frac{i(a(x,i)-1)}{N}$ denotes the probability of forming a $2i$-cluster from any two i-clusters among all i-clusters $(a(x,i))$ in site x. Note that $a(x,i)(a(x,i) - 1)/2$ is the number of ways of forming a $2i$-cluster from any two i-clusters among all i-clusters $(a(x,i))$ in site x.

3. The stationary distribution

The stationary distributions for the homogeneous random coagulation-fragmentation processes have been given by Han [10] and Durrett *et al* [7]. Here we shall present the stationary distribution for the HCFP. Assume the following

H_1: the diffusion rate $g(.)$ satisfies

$$\sup_m |g(m+1) - g(m)| < \infty.$$

H_2:

$$K_{ij} = K_{ji}, \qquad F_{ij} = F_{ji}, \qquad K_{ij}h(i)h(j) = \lambda F_{ij}h(i+j), \qquad i, j \geqslant 0, \tag{1}$$

where $\frac{1}{\lambda}(\lambda > 0)$ represents the fragmentation strength and $h(.)$ is a positive function. As Van Dongen and Ernst [21] stated, when the process describes the system of polymers in which intramolecular reactions do not occur, and therefore only branched-chain (non-cyclic) polymers are formed and all unreacted functional groups are equally reactive, $k!h(k)$ may denote the number of distinct ways of forming k-mers from k distinguishable units and equation (1) states that the number of distinct ways for $(i + j)$-mers to break up into i-mer and j-mers $(\lambda F(i,j)h(i+j))$ equals the number of bonds between (i) and (j) polymers in $(i + j)$-mer configurations $(K(i,j)h(i)h(j))$. In fact, the total fragmentation rate of a k-mer is taken to be proportional to the number of bonds in [21], i.e.

$$\frac{1}{2}\sum_{i+j=k} F_{ij} = \frac{1}{\lambda}(k-1).$$

In particular all bonds are equally breakable, and the total rate of k-mer break up, $\frac{1}{2}\sum F_{ij}$, is proportional to the number of bonds. Hence, by equation (1) we have

$$(k-1)h(k) = \frac{1}{2}\sum_{i+j=k} K_{ij}h(i)h(j).$$

The quantities w_k defined by $w_k = k!h(k)$ represent the number of distinct ways in which a k-mer can be constructed out of k monomeric units, assuming that units and functional groups are distinguishable. This can be seen as follows. By the definition of w_k, we have

$$(k-1)w_k = \frac{1}{2}\sum_{i+j=k}\frac{k!}{i!j!}K_{ij}w_i w_j.$$

According to the right-hand side of the above equation, one may choose i units out of k (distinguishable) units in C_k^i different ways in order to build i- and j-mers (which may be constructed in $w_i w_j$ different ways). Since functional groups are also distinguishable, such i- and j-mers may be joined in K_{ij} ways. Equation (1) is usually called a detailed balance condition.

Note that the condition H_1 is not necessary for obtaining the stationary distribution in the following, but it can guarantee that the limit process ($N \to \infty$ and $B \nearrow \mathbb{Z}^d$) of $\{A_N(t), t \geqslant 0\}$ is a unique Feller process (see [12]).

Theorem 1. *Suppose the two conditions H_1 and H_2 hold. Then $\{A_N(t), t \geqslant 0\}$ has a unique stationary distribution μ_N given by*

$$\mu_N(A) = \frac{1}{Z_N}\prod_{x\in B}\prod_{k=1}^{N}\frac{\left[\frac{N}{\lambda}h(k)\right]^{a(x,k)}}{g(a(x,k))!}, \qquad A \in \mathbb{X}_N(B), \tag{2}$$

and the process is reversible with this stationary distribution, where Z_N is the normalization factor, i.e.

$$Z_N = \sum_{A\in\mathbb{X}_N(B)}\prod_{x\in B}\prod_{k=1}^{N}\frac{\left[\frac{N}{\lambda}h(k)\right]^{a(x,k)}}{g(a(x,k))!}, \tag{3}$$

where $g(m)! = g(1)g(2)\cdots g(m)$ with $g(0)! := 1$. Usually, Z_N is called the partition function of the process.

Proof. We first check that

$$\mu_N(A')Q_{A'A} = \mu_N(A)Q_{AA'} \tag{4}$$

for all $A, A' \in \mathbb{X}_N(B)$. It is equivalent to check

$$\frac{Q(A,A')}{Q(A',A)} = \frac{\mu(A')}{\mu(A)}$$

for all $A, A' \in \mathbb{X}_N(B)$. In fact, we have

$$\frac{Q(A,A')}{Q(A',A)} = \frac{g(a(x,k))}{g(a(y,k)+1)} = \frac{g(a(x,k))}{\frac{N}{\lambda}h(k)}\frac{\frac{N}{\lambda}h(k)}{g(a(y,k)+1)} = \frac{\mu(A')}{\mu(A)}$$

for the case $A' = A_{x,y}^k$ and, by (1),

$$\begin{aligned}\frac{Q(A,A')}{Q(A',A)} &= \frac{K_{ij}}{NF_{ij}}\frac{g(a(x,i))g(a(x,j))}{g(a(x,i+j)+1)}\\ &= \frac{\lambda h(i+j)}{Nh(i)h(j)}\frac{g(a(x,i))g(a(x,j))}{g(a(x,i+j)+1)}\\ &= \frac{\frac{N}{\lambda}h(i+j)}{g(a(x,i+j)+1)}\frac{g(a(x,i))}{\frac{N}{\lambda}h(i)}\frac{g(a(x,j))}{\frac{N}{\lambda}h(j)}\\ &= \frac{\mu(A')}{\mu(A)}\end{aligned}$$

for $A' = A^+_{x,ij}, i \neq j$. Similarly, we can check that (4) holds for $A' = A^+_{x,ij}, i = j$, and $A' = A^-_{x,ij}$. Thus, (4) holds for all $A, A' \in \mathbb{X}_N(B)$, and therefore

$$\sum_{A'\in\mathbb{X}_N(B)} \mu_N(A')Q_{A'A} = \mu_N(A) \sum_{A'\in\mathbb{X}_N(B)} Q_{AA'} = 0.$$

This means that μ_N is a reversible stationary distribution of the process. Since all states in $\mathbb{X}_N(B)$ connect mutually, that is, for $A, A' \in \mathbb{X}_N(B)$, there are $A_1, A_2, \ldots, A_k \in \mathbb{X}_N(B)$ $(k \geqslant 1)$ such that $Q(A, A_1)Q(A_1, A_2)\ldots Q(A_k, A') > 0$. This means that the process is an irreducible Markov chain on the finite state space, so the stationary distribution is unique. □

4. Asymptotic distributions of the clusters

In this section, we shall study the asymptotic behavior of the process. The convergence considered here is to converge in distribution. We use $E_N(.)$ and $E(.)$ to denote, respectively, the expectation corresponding to the stationary probability distribution μ_N and the limit of $E_N(.)$, i.e. $E(.) = \lim_{N\to\infty} E_N(.)$. The methods and techniques used in the proofs of theorems come mainly from [10, 11, 18, 19].

Now we assume that the diffusion rate $g(.)$ satisfies

$$g(k) = \gamma k, \tag{5}$$

where $\gamma > 0$ denotes the diffusion strength. It follows from (2) and (5) that

$$\mu_N(A) = \frac{1}{Z_N} \prod_{x\in B} \prod_{k=1}^{N} \frac{\left[\frac{N}{\lambda\gamma} h(k)\right]^{a(x,k)}}{a(x,k)!}, \qquad A \in \mathbb{X}_N(B). \tag{6}$$

Let the following series $F(u)$ has a positive radius, r, of convergence and

$$F(u) = \sum_{k=1}^{\infty} h(k)u^k < \infty \tag{7}$$

for $|u| \leqslant r$. Let

$$Z(N, y) = \sum_{A\in\mathbb{X}_N(B)} \prod_{x\in B} \prod_{k=1}^{N} \frac{[yh(k)]^{a(x,k)}}{a(x,k)!}$$

for $y > 0$. Next, an integral expression of the partition function Z_N will be given in lemma 1. We shall use $a(x, j)$ to denote a value of random variable $\mathbf{a}(\mathbf{x}, \mathbf{j})$ in the following.

Lemma 1. *Suppose the three conditions (1), (5) and (7) hold. Then*

$$Z_N = Z\left(N, \frac{N}{\lambda\gamma}\right) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \exp\left\{\frac{|B|N}{\lambda\gamma} F(u)\right\} \frac{\mathrm{d}u}{u^{N+1}}, \tag{8}$$

$$E_N[\mathbf{a}(\mathbf{x}, \mathbf{k})] = \frac{N}{\lambda\gamma} h(k) \frac{Z\left(N-k, \frac{N}{\lambda\gamma}\right)}{Z\left(N, \frac{N}{\lambda\gamma}\right)}, \qquad x \in B, \tag{9}$$

and

$$E_N[\mathbf{a}(\mathbf{x}, \mathbf{k})^2] = \left[\frac{N}{\lambda\gamma} h(k)\right]^2 \frac{Z\left(N-2k, \frac{N}{\lambda\gamma}\right)}{Z\left(N, \frac{N}{\lambda\gamma}\right)} + \frac{N}{\lambda\gamma} h(k) \frac{Z\left(N-k, \frac{N}{\lambda\gamma}\right)}{Z\left(N, \frac{N}{\lambda\gamma}\right)}, \qquad x \in B, \tag{10}$$

where Γ is a contour with its radius equal to r surrounding the origin $u = 0$.

Proof. Let $B = \{x_1, \ldots, x_m\}$ and $A = (a(x_j, k), 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant N) \in \mathbb{X}_N(B)$, where $\sum_{j=1}^{m}\sum_{k=1}^{N} ka(x_j, k) = N$. Assume that $a(x'_j, k') \geqslant 1$ for some x'_j and k'. Let $A' = (a'(x_j, k), 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant N)$, where $a'(x_j, k)$ are defined by $a'(x_j, k) = a(x'_j, k') - 1$ for $x_j = x'_j$ and $k = k'$; otherwise $a'(x_j, k) = a(x_j, k)$. Thus, $A' = (a'(x_j, k), 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant N) \in \mathbb{X}_{N-k'}(B)$ since $\sum_{j=1}^{m}\sum_{k=1}^{N} ka'(x_j, k) = N - k' = \sum_{j=1}^{m}\sum_{k=1}^{N-k'} ka'(x_j, k)$. We shall use this fact in the following computation. Since

$$Z(N, y) = \sum_{A\in\mathbb{X}_N(B)} \prod_{j=1}^{m}\prod_{k=1}^{N} \frac{[yh(k)]^{a(x_j,k)}}{a(x_j, k)!} = \sum_{A\in\mathbb{X}_N(B)} y^{\sum_{j=1}^{m}\sum_{k=1}^{N} a(x_j,k)} \prod_{j=1}^{m}\prod_{k=1}^{N} \frac{h(k)^{a(x_j,k)}}{a(x_j, k)!},$$

it follows that

$$\begin{aligned}
Z'_y(N, y) &= \sum_{A\in\mathbb{X}_N(B)} \sum_{j=1}^{m}\sum_{k=1}^{N} a(x_j, k) y^{\sum_{j=1}^{m}\sum_{k=1}^{N} a(x_j,k)-1} \prod_{j=1}^{m}\prod_{k=1}^{N} \frac{h(k)^{a(x_j,k)}}{a(x_j, k)!} \\
&= \sum_{A\in\mathbb{X}_N(B)} \left\{ \sum_{j=1}^{m}\sum_{k=1}^{N} a(x_j, k) y^{a(x_j,k)-1} \frac{h(k)^{a(x_j,k)}}{a(x_j, k)!} \right. \\
&\qquad \left. \times \prod_{l\neq k} \frac{[yh(l)]^{a(x_j,l)}}{a(x_j, l)!} \prod_{i\neq j}\prod_{l=1}^{N} \frac{[yh(l)]^{a(x_i,l)}}{a(x_i, l)!} \right\} \\
&= \sum_{A\in\mathbb{X}_N(B)} \left\{ \sum_{j=1}^{m}\sum_{k=1}^{N} h(k) y^{a(x_j,k)-1} \frac{h(k)^{a(x_j,k)-1}}{(a(x_j, k)-1)!} \right. \\
&\qquad \left. \times \prod_{l\neq k} \frac{[yh(l)]^{a(x_j,l)}}{a(x_j, l)!} \prod_{i\neq j}\prod_{l=1}^{N} \frac{[yh(l)]^{a(x_i,l)}}{a(x_i, l)!} \right\} \\
&= \sum_{k=1}^{N} h(k) \sum_{A\in\mathbb{X}_{N-k}(B)} \left\{ \sum_{j=1}^{m}\prod_{l=1}^{N} \frac{[yh(l)]^{a'(x_j,l)}}{a'(x_j, l)!} \prod_{i\neq j}\prod_{l=1}^{N} \frac{[yh(l)]^{a(x_i,l)}}{a(x_i, l)!} \right\} \\
&= \sum_{k=1}^{N} h(k) \sum_{A\in\mathbb{X}_{N-k}(B)} \left\{ \sum_{j=1}^{m}\prod_{l=1}^{N-k} \frac{[yh(l)]^{a(x_j,l)}}{a(x_j, l)!} \prod_{i\neq j}\prod_{l=1}^{N-k} \frac{[yh(l)]^{a(x_i,l)}}{a(x_i, l)!} \right\} \\
&= m\sum_{k=1}^{N} h(k) \sum_{A\in\mathbb{X}_{N-k}(B)} \prod_{i=1}^{m}\prod_{l=1}^{N-k} \frac{[yh(l)]^{a(x_i,l)}}{a(x_i, l)!} \\
&= |B| \sum_{k=1}^{N} h(k) Z(N-k, y),
\end{aligned}$$

where we define $\sum_{A:|A|=0}\prod_{k=1}^{0} = 1$. We now prove that the differentiation w.r.t. the variable y can be done in the infinite sum $I(u, y)$ for $u \leqslant r$. □

We only consider the case: $B = \{x\}$ and $|A_x| = N$. We can similarly discuss this for $|B| > 0$. Since

$$F(u) = \sum_{k=1}^{\infty} h(k)u^k = \sum_{k=1}^{\infty} [(h(k))^{\frac{1}{k}} u]^k < \infty \tag{11}$$

for $|u| \leqslant r$, it follows that $[(h(k))^{1/k} r]^k \to 0$ as $k \to \infty$. Hence, there exists a positive number k_0 such that $(h(k))^{1/k} r < 1$ for $k \geqslant k_0$. Note that, for any fixed $\bar{y}$ $(0 < \bar{y} < \infty)$,

$(\bar{y})^{1/k} \to 1$ as $k \to \infty$. We can take a positive number k_0 such that $(\bar{y}h(k))^{1/k} r \leqslant c < 1$ for $k \geqslant k_0$. Let $n_k = a(x,k)$. Note that $\sum_{k=1}^{N} kn_k = N$ and $n_k = 0$ for $k > k_0$ when $\sum_{k=1}^{k_0} kn_k \geqslant N - k_0$, where $N > k_0$. Thus, there are two positive numbers $a_0 > 1$ and $b_0 > 1$ which depend only on k_0 such that

$$
\begin{aligned}
Z(N,y)u^N &= \sum_{A \in \mathbb{X}_N(\{x\})} \prod_{k=1}^{N} \frac{[yh(k)]^{a(x,k)}}{a(x,k)!} u^N \\
&= \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{N} kn_k = N} \prod_{k=1}^{k_0} \frac{[yh(k)u^k]^{n_k}}{n_k!} \prod_{k=k_0+1}^{N} \frac{[(yh(k))^{1/k}u]^{kn_k}}{n_k!} \\
&\leqslant \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{N} kn_k = N} \prod_{k=1}^{k_0} \frac{[\bar{y}h(k)u^k]^{n_k}}{n_k!} \frac{c^{N-\sum_{k=1}^{k_0} kn_k}}{n_{k_0+1}!\cdots n_N!} \\
&\leqslant \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k \geqslant N-k_0} \prod_{k=1}^{k_0} \frac{[(\bar{y}h(k))^{1/k}r]^{kn_k}}{n_k!} \\
&\quad + \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k < N-k_0} \prod_{k=1}^{k_0} \frac{[(\bar{y}h(k))^{1/k}r]^{kn_k}}{n_k!} \frac{c^{N-\sum_{k=1}^{k_0} kn_k}}{n_{k_0+1}!\cdots n_N!} \\
&\leqslant \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k \geqslant N-k_0} \prod_{k=1}^{k_0} \frac{k!(n_k)^k[(\bar{y}h(k))^{1/k}r]^{kn_k}}{(kn_k)!} \\
&\quad + \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k < N-k_0} \prod_{k=1}^{k_0} \frac{k!(n_k)^k[(\bar{y}h(k))^{1/k}r]^{kn_k}}{(kn_k)!} \frac{c^{N-\sum_{k=1}^{k_0} kn_k}}{n_{k_0+1}!\cdots n_N!} \\
&\leqslant (k_0!)^{k_0} \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k \geqslant N-k_0} \prod_{k=1}^{k_0} \frac{\left[(n_k+1)^{\frac{1}{n_k+1}}(\bar{y}h(k))^{1/k}r\right]^{kn_k}}{(kn_k)!} \\
&\quad + (k_0!)^{k_0} \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k < N-k_0} \prod_{k=1}^{k_0} \frac{\left[(n_k+1)^{\frac{1}{n_k+1}}(\bar{y}h(k))^{1/k}r\right]^{kn_k}}{(kn_k)!} \frac{c^{N-\sum_{k=1}^{k_0} kn_k}}{n_{k_0+1}!\cdots n_N!} \\
&\leqslant (k_0!)^{k_0} \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k \geqslant N-k_0} \prod_{k=1}^{k_0} \frac{[2(\bar{y}h(k))^{1/k}r]^{kn_k}}{(kn_k)!} \\
&\quad + (k_0!)^{k_0} \sum_{(n_1,\ldots,n_N):\sum_{k=1}^{k_0} kn_k < N-k_0} \prod_{k=1}^{k_0} \frac{[2(\bar{y}h(k))^{1/k}r]^{kn_k}}{(kn_k)!} \frac{c^{N-\sum_{k=1}^{k_0} kn_k}}{n_{k_0+1}!\cdots n_N!} \\
&\leqslant a_0(k_0!)^{k_0} \frac{\left[2r\sum_{k=1}^{k_0}(\bar{y}h(k))^{1/k}\right]^{N-k_0}}{(N-k_0)!} \\
&\quad + b_0(k_0!)^{k_0} N \sum_{m=1}^{N-k_0-1} \frac{\left[2r\sum_{k=1}^{k_0}(\bar{y}h(k))^{1/k}\right]^{m}}{m!} c^{N-m} \\
&\leqslant a_0(k_0!)^{k_0} \frac{\left[2r\sum_{k=1}^{k_0}(\bar{y}h(k))^{1/k}\right]^{N-k_0}}{(N-k_0)!}
\end{aligned}
$$

$$+ \; b_0(k_0!)^{k_0} N \left\{ \sum_{m=1}^{N-k_0-1} \frac{\left[2r/c \sum_{k=1}^{k_0} (\bar{y}h(k))^{1/k}\right]^m}{m!} \right\} c^N$$

$$\leqslant a_0(k_0!)^{k_0} \frac{[A(r,\bar{y},k_0)]^{N-k_0}}{(N-k_0)!} + b_0(k_0!)^{k_0} N \exp\{A(r,\bar{y},k_0)/c\} c^N$$

for $N > k_0$, $0 \leqslant y \leqslant \bar{y}$ and $0 \leqslant u \leqslant r$, where $A(r,\bar{y},k_0) = 2r\sum_{k=1}^{k_0}(\bar{y}h(k))^{1/k}$. Hence,

$$I'_y(u,y) = \sum_{N=1}^{\infty} Z'_y(N,y)u^N = \sum_{N=1}^{\infty}\sum_{j=1}^{N} h(j)Z(N-j,y)u^N$$

$$= \sum_{j=1}^{\infty} h(j)u^j \sum_{N-j=1}^{\infty} Z(N-j,y)u^{(N-j)}$$

$$\leqslant F(u)\left[\sum_{n=1}^{k_0} Z(n,y)u^n + \sum_{n=k_0+1}^{\infty}\left[a_0(k_0!)^{k_0}\frac{[A(r,\bar{y},k_0)]^{n-k_0}}{(n-k_0)!} + b_0(k_0!)^{k_0} n \exp\{A(r,\bar{y},k_0)/c\}c^n\right]\right] < \infty.$$

for $0 \leqslant y \leqslant \bar{y}$ and $0 \leqslant u \leqslant r$. This means that the differentiation w.r.t. the variable y can be done in the infinite sum for $u \leqslant r$.

Hence

$$I'_y(u,y) = \sum_{N=1}^{\infty} Z'_y(N,y)u^N = |B|\sum_{N=1}^{\infty}\sum_{j=1}^{N} h(j)Z(N-j,y)u^N$$

$$= |B|\sum_{j=1}^{\infty} h(j)u^j \sum_{N-j=1}^{\infty} Z(N-j,y)u^{(N-j)} = |B|F(u)I(u,y),$$

and therefore

$$I(u,y) = \exp\{y|B|F(u)\}. \tag{12}$$

By using the Cauchy integral formula for (11) and taking $y = N/(\lambda\gamma)$, we can obtain (8).

It follows from (6) that

$$E_N[\mathbf{a}(\mathbf{x},\mathbf{k})] = \frac{1}{Z_N}\sum_{A\in\mathbb{X}_N(B)} a(x,k)\frac{\left[\frac{N}{\lambda\gamma}h(k)\right]^{a(x,k)}}{a(x,k)!}\prod_{j\neq k}^{N}\frac{\left[\frac{N}{\lambda\gamma}h(j)\right]^{a(x,j)}}{a(x,j)!}\prod_{z\neq x}\prod_{k=1}^{N}\frac{\left[\frac{N}{\lambda\gamma}h(k)\right]^{a(z,k)}}{a(z,k)!}$$

$$= \frac{N}{\lambda\gamma}h(k)\frac{1}{Z_N}\sum_{A\in\mathbb{X}_{N-k}(B)}\prod_{j=1}^{N-k}\frac{\left[\frac{N}{\lambda\gamma}h(j)\right]^{a(x,j)}}{a(x,j)!}\prod_{z\neq x}\prod_{k=1}^{N}\frac{\left[\frac{N}{\lambda\gamma}h(k)\right]^{a(z,k)}}{a(z,k)!}$$

$$= \frac{N}{\lambda\gamma}h(k)\frac{Z\left(N-k,\frac{N}{\lambda\gamma}\right)}{Z\left(N,\frac{N}{\lambda\gamma}\right)}.$$

This is (9). The equality (10) can be similarly obtained.

As we know that if $N \to \infty$, then at least one of $|A_x|, x \in B$ goes to ∞ since $N = \sum_{x\in B}|A_x|$ and B is a finite set. For fixed $|B|$, let $\rho = \lambda\gamma/|B|$. Obviously, the fragmentation and diffusion strengths, λ and γ, can form different hyperbola for different values of ρ. Next, we give a definition of a gelation in the HCFP in order to study the critical behavior of the process.

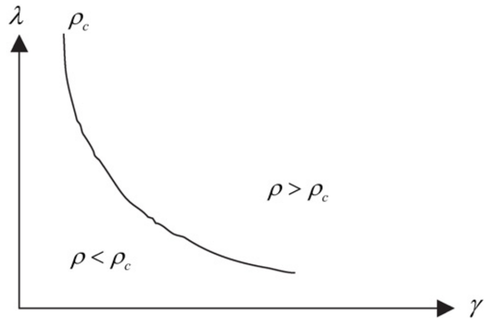

Figure 1. A critical curve.

Definition 1. *We say that there is a gelation in the HCFP, if there is a critical curve* ρ_c *such that*

$$\lim_{N\to\infty} \frac{1}{N} E_N \left[\sum_{x\in B} \sum_{k=1}^{N} k\mathbf{a}(\mathbf{x}, \mathbf{k}) \right] = 1$$

for $\rho \leqslant \rho_c$ *and*

$$\lim_{N\to\infty} \frac{1}{N} E_N \left[\sum_{x\in B} \sum_{k=1}^{N} k\mathbf{a}(\mathbf{x}, \mathbf{k}) \right] < 1$$

for $\rho > \rho_c$.

By theorem 2 of [11], we know that if

$$h(k) = (1 + o(1))c\tilde{r}^{-k}k^{-\beta}, \tag{13}$$

where $c, \tilde{r}$ and β are three positive constants, then there is a gelation in the homogeneous coagulation-fragmentation process when $2 < \beta < 3$. It can be checked that the numbers $h(k), k \geqslant 1$, for many models, such as RA_a $(a \geqslant 3)$, $RA_\infty, A_a RB_b$ $(\min(a, b) \geqslant 2)$, $A_a RB_\infty$ $(a \geqslant 2)$, etc. satisfy (12) (see [11, 21]).

So, we shall assume that $h(k)$ satisfies (12) and $2 < \beta < 3$ in the following. Thus, (7) holds for $|u| \leqslant \tilde{r}$ and

$$F'(\tilde{r}) = \lim_{u\to\tilde{r}-0} F'(u) < +\infty; \qquad F''(\tilde{r}) = \lim_{u\to\tilde{r}-0} F''(u) = +\infty.$$

By using (6), (8), (9), (12) and theorem 2 of [11], we know that a critical curve ρ_c for the occurrence of gelation in the HCFP can be determined by

$$\rho_c = \tilde{r}F'(\tilde{r}).$$

We usually call $\rho < \rho_c, \rho = \rho_c$ and $\rho > \rho_c$ as the sub-critical, critical and super-critical stages, respectively (see figure 1).

Let $\rho < \rho_c$ and $D_N(u) = \frac{N}{\rho}F(u) - N\log u$ for $0 < u \leqslant \tilde{r}$. Let r be a root of equation $D'_N(u) = 0$. Then r satisfies $\rho = rF'(r)$ and $r < \tilde{r}$ for all N. Note that $F'(u)$ is a strictly monotone increasing function on $[0, \tilde{r}]$, and therefore r is a saddle point of $\exp\{D_N(u)\}$. By the standard saddle-point-type argument (see [20], p 96) and (8), we can get

$$Z_N = Z\left(N, \frac{N}{\lambda\gamma}\right) = (1 + o(1))\frac{1}{\sqrt{2\pi A(r)N}} \exp\{D_N(r)\}, \tag{14}$$

where

$$A(r) = \frac{rF'(r) + r^2F''(r)}{\rho}. \tag{15}$$

Note that

$$\int_{-\infty}^{+\infty} \exp\{ibx - a^2x^2\}\,dx = \frac{\sqrt{\pi}}{a}\exp\left\{-\frac{b^2}{4a^2}\right\},$$

where $a > 0$ and $i = \sqrt{-1}$. We can similarly obtain

$$Z\left(N-k, \frac{N}{\lambda\gamma}\right) = (1+o(1))\frac{r^k}{\sqrt{2\pi A(r)N}}\exp\left\{-\frac{k^2}{4A(r)N}\right\}\exp\{D_N(r)\}.$$

Hence,

$$Z\left(N-k, \frac{N}{\lambda\gamma}\right)\Big/ Z\left(N, \frac{N}{\lambda\gamma}\right) = (1+o(1))r^k\exp\left\{-\frac{k^2}{4A(r)N}\right\}. \tag{16}$$

It follows from (9), (10) and (15) that the expectation and variance of $\mathbf{a}(\mathbf{x}, \mathbf{k})$ satisfy

$$E_N[\mathbf{a}(\mathbf{x}, \mathbf{k})] = Var_N[\mathbf{a}(\mathbf{x}, \mathbf{k})] = (1+o(1))\frac{N}{\rho|B|}h(k)r^k \tag{17}$$

for large N.

Let $\mathbf{a}^*(x, j)(r) = [\mathbf{a}(\mathbf{x}, \mathbf{j}) - Na_j(r)]/\sqrt{Na_j(r)})$, where $a_j(r) = h(j)r^j/(\rho|B|), 0 < r \leqslant \tilde{r}$. In fact, $Na_j(r)$ and $\sqrt{Na_j(r)}$ are the expectation and standard variance of $\mathbf{a}(\mathbf{x}, \mathbf{j})$ respectively.

Theorem 2. *Suppose the three conditions (1), (5) and (12) hold.*

(i) If $\rho < \rho_c$ and r satisfies $\rho = rF'(r)$, then $\{\mathbf{a}^*(x, j)(r), x \in B, j \geqslant 1\}$ converges to a mutually dependent Gaussian sequence $\{G_j(x), x \in B, j \geqslant 1\}$ as $N \to \infty$ with

$$E[G_j(x)] = 0, \qquad E[G_j(x)G_l(y)] = \delta_{jl}(xy) - g_j g_l$$

and

$$g_j = \frac{j\sqrt{h(j)r^j}}{\sqrt{|B|}\sqrt{rF'(r) + r^2F''(r)}},$$

where $\delta_{jl}(xy) = 0$ for $j \neq l$ or $x \neq y$ and $\delta_{jl}(xy) = 1$ for $j = l$ and $x = y$.

(ii) If $\rho \geqslant \rho_c$, then $\{\mathbf{a}^*(x, j)(\tilde{r}), x \in B, j \geqslant 1\}$ converges to a mutually independent Gaussian sequence $\{G_j(x), x \in B, j \geqslant 1\}$ as $N \to \infty$ with

$$E[G_j] = 0, \qquad E[G_j(x)G_l(y)] = \delta_{jl}(xy).$$

Proof. Let $B = \{x_1, \ldots, x_m\}, T_k = \{(t_1(x), \ldots, t_k(x), 0, \ldots) : x \in B\}$ and $T(x) = (t_1(x), \ldots, t_n(x), \ldots)$ for $x \in B$, where $t_j(x), x \in B$, are all real numbers. Denote by $\Phi_N(T_k)$ $(k \leqslant N)$ the characteristic function of random variables $\mathbf{a}(\mathbf{x}, \mathbf{j}), 1 \leqslant j \leqslant k, x \in B$. Then

$$\begin{aligned}\Phi_N(T_N) &= E_N\left[\prod_{x\in B}\prod_{j=1}^{N} e^{it_j(x_l)\mathbf{a}(\mathbf{x},\mathbf{j})}\right]\\ &= \sum_{A\in\mathbb{X}_N(B)}\prod_{l=1}^{m}\prod_{j=1}^{N} e^{it_j(x_l)a(x_l,j)}\mu_N(A)\\ &= \frac{1}{Z_N}\sum_{A\in\mathbb{X}_N(B)}\prod_{l=1}^{m}\prod_{j=1}^{N}\frac{\left[\frac{N}{\lambda\gamma}e^{it_j(x_l)}h(j)\right]^{a(x_l,j)}}{a(x_l,j)!},\end{aligned} \tag{18}$$

and therefore

$$Z_N(y)\Phi_N(T_N) = \sum_{A\in\mathbb{X}_N(B)} \prod_{l=1}^{m}\prod_{j=1}^{N} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_l)}h(j)]^{a(x_l,j)}}{a(x_l,j)!},$$

where $i = \sqrt{-1}$ and $y > 0$. Let $g(y,N) = Z_N(y)\Phi_N(T_N)$ and $G(y,u) = 1 + \sum_{N=1}^{\infty} g(y,N)u^N$, we have

$$\begin{aligned}
\frac{\partial g(y,N)}{\partial y} &= \sum_{A\in\mathbb{X}_N(B)} \left\{ \sum_{l=1}^{m}\sum_{k=1}^{N} a(x_l,k)y^{a(x_l,k)-1}\frac{[\mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)]^{a(x_l,k)}}{(a(x_l,k)!} \prod_{j\neq k}^{N} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_l)}h(j)]^{a(x_l,j)}}{a(x_l,j)!} \right. \\
&\quad \left. \times \prod_{i\neq l}\prod_{j=1}^{N} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_i)}h(j)]^{a(x_i,j)}}{a(x_i,j)!} \right\} \\
&= \sum_{A\in\mathbb{X}_N(B)} \left\{ \sum_{l=1}^{m}\sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)\frac{[y\,\mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)]^{a(x_l,k)-1}}{(a(x_l,k)-1)!} \prod_{j\neq k}^{N} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_l)}h(j)]^{a(x_l,j)}}{a(x_l,j)!} \right. \\
&\quad \left. \times \prod_{i\neq l}\prod_{j=1}^{N} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_i)}h(j)]^{a(x_i,j)}}{a(x_i,j)!} \right\} \\
&= \sum_{k=1}^{N}\sum_{A\in\mathbb{X}_{N-k}(B)} \left\{ \sum_{l=1}^{m} \mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)\frac{[y\,\mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)]^{a(x_l,k)-1}}{(a(x_l,k)-1)!} \prod_{j=1}^{N-k} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_l)}h(j)]^{a(x_l,j)}}{a(x_l,j)!} \right. \\
&\quad \left. \times \prod_{i\neq l}\prod_{j=1}^{N-k} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_i)}h(j)]^{a(x_i,j)}}{a(x_i,j)!} \right\} \\
&= \sum_{k=1}^{N}\sum_{A\in\mathbb{X}_{N-k}(B)} \left\{ \sum_{l=1}^{m} \mathrm{e}^{\mathrm{i}t_k(x_l)}h(k) \prod_{j=1}^{N-k} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_l)}h(j)]^{a(x_l,j)}}{a(x_l,j)!} \right. \\
&\quad \left. \times \prod_{i\neq l}\prod_{j=1}^{N-k} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_i)}h(j)]^{a(x_i,j)}}{a(x_i,j)!} \right\} \\
&= \sum_{k=1}^{N}\sum_{A\in\mathbb{X}_{N-k}(B)} \left\{ \sum_{l=1}^{m} \mathrm{e}^{\mathrm{i}t_k(x_l)}h(k) \prod_{i=1}^{m}\prod_{j=1}^{N-k} \frac{[y\,\mathrm{e}^{\mathrm{i}t_j(x_i)}h(j)]^{a(x_i,j)}}{a(x_i,j)!} \right\} \\
&= \sum_{k=1}^{N}\sum_{l=1}^{m} \mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)Z_{N-k}(y)\Phi_N(T_{N-k}) \\
&= \sum_{l=1}^{m}\sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)g(y,N-k).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial G(y,N)}{\partial y} &= \sum_{N=1}^{\infty} \frac{\partial g(y,N)}{\partial y}u^N \\
&= \sum_{N=1}^{\infty}\sum_{l=1}^{m}\sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}t_k(x_l)}h(k)u^k g(y,N-k)u^{N-k} \\
&= \sum_{l=1}^{m} F(T(x_l),u)G(y,u),
\end{aligned}$$

where $F(T(x_l), u) = \sum_{k=1}^{\infty} \mathrm{e}^{\mathrm{i}t_k(x_l)} h(k) u^k$. Hence $G(y, u) = \mathrm{e}^{y \sum_{x \in B} F(T(x), u)}$ and, by Cauchy's integral formula,

$$Z_N(y)\Phi_N(T_N) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \exp\left\{ y \sum_{x \in B} F(T(x), u) \right\} \frac{\mathrm{d}u}{u^{N+1}}.$$

Taking $y = N/(\lambda\gamma)$, we have

$$\Phi_N(T_N) = \frac{1}{2\pi \mathrm{i} Z_N} \int_\Gamma \exp\left\{ \frac{N}{\lambda\gamma} \sum_{x \in B} F(T(x), u) \right\} \frac{\mathrm{d}u}{u^{N+1}}. \tag{19}$$

□

To prove (i), let $N > k$ for fixed k and let $t_l(x) = 0$ for all $l > k$ and $x \in B$. Then

$$F(T(x), u) = \sum_{j=1}^{k} [\mathrm{e}^{\mathrm{i}t_j(x)} - 1] h(j) u^j + F(u), \qquad x \in B. \tag{20}$$

Taking $u = r\,\mathrm{e}^{\mathrm{i}\theta}$, it follows from (17) and (18) that

$$\begin{aligned}
\Phi_N(T_k) &= \Phi_N(\{(t_1(x), \ldots, t_k(x), 0, \ldots) : x \in B\}) \\
&= \frac{1}{2\pi \mathrm{i} Z_N} \int_\Gamma \exp\left\{ \frac{N}{\rho} F(u) + \frac{N}{\rho|B|} \sum_{x \in B} \sum_{j=1}^{k} [\mathrm{e}^{\mathrm{i}t_j(x)} - 1] h(j) u^j] \right\} u^{-(N+1)}\, \mathrm{d}u \\
&= \frac{\exp\{D_N(r)\}}{2\pi Z_N} \int_{-\pi}^{\pi} \exp\left\{ \frac{N}{\rho} [F(r\,\mathrm{e}^{\mathrm{i}\theta}) - F(r)] - \mathrm{i}N\theta \right. \\
&\quad \left. + \frac{N}{\rho|B|} \sum_{x \in B} \sum_{j=1}^{k} [\mathrm{e}^{\mathrm{i}t_j(x)} - 1] h(j) r^j \mathrm{e}^{\mathrm{i}(j\theta)} \right\} \mathrm{d}\theta.
\end{aligned}$$

Since

$$[F(r\,\mathrm{e}^{\mathrm{i}\theta}) - F(r)]/\rho = \mathrm{i}\theta - \tfrac{1}{2} A(r)\theta^2 + o(\theta^2)$$

and

$$\mathrm{e}^{\mathrm{i}u} - 1 = \mathrm{i}u - \tfrac{1}{2}u^2 + o(u^2),$$

taking $\sqrt{N}\theta = s$ and $t_j(x) = t'_j(x)/\sqrt{N a_j(r)}$, we have

$$\begin{aligned}
&\frac{N}{\rho}[F(r\,\mathrm{e}^{\mathrm{i}\theta}) - F(r)] - \mathrm{i}N\theta + \frac{N}{\rho|B|} \sum_{x \in B} \sum_{j=1}^{k} [\mathrm{e}^{\mathrm{i}t_j(x)} - 1] h(j) r^j \mathrm{e}^{\mathrm{i}(j\theta)} \\
&\qquad = \frac{1}{2} A(r) s^2 - C(r) s + M(r) + o(1)
\end{aligned}$$

for large N, where $A(r)$ is defined in (14),

$$C(r) = \sum_{x \in B} \sum_{j=1}^{k} j \sqrt{a_j(r)}\, t'_j(x)$$

and

$$M(r) = \mathrm{i}\sqrt{N} \sum_{x \in B} \sum_{j=1}^{k} \sqrt{a_j(r)}\, t'_j(x) - \frac{1}{2} \sum_{x \in B} \sum_{j=1}^{k} (t'_j(x))^2.$$

Hence,

$$\begin{aligned}
&\Phi_N\left(\left\{\left(\frac{t_1'(x)}{\sqrt{Na_1(r)}},\ldots,\frac{t_k'(x)}{\sqrt{Na_k(r)}},0,\ldots\right):x\in B\right\}\right)\\
&\quad=\Phi_N(\{(t_1(x),\ldots,t_k(x),0,\ldots):x\in B\})\\
&\quad=\frac{\exp\{D_N(r)+M(r)\}}{2\pi\sqrt{N}Z_N}\int_{-\pi\sqrt{N}}^{\pi\sqrt{N}}\exp\left\{-\frac{1}{2}A(r)s^2-C(r)s+o(1)\right\}\mathrm{d}s\\
&\quad=\frac{\exp\{D_N(r)+M(r)\}}{2\pi\sqrt{N}Z_N}\int_{-\pi\sqrt{N}}^{\pi\sqrt{N}}\exp\left\{-\frac{A(r)}{2}\left(s+\frac{C(r)}{A(r)}\right)^2+\frac{C^2(r)}{2A(r)}+o(1)\right\}\mathrm{d}s\\
&\quad=(1+o(1))\frac{\exp\{D_N(r)\}}{\sqrt{2\pi A(r)}\sqrt{N}Z_N}\exp\left\{M(r)+\frac{C^2(r)}{2A(r)}\right\}\\
&\quad=(1+o(1))\exp\left\{M(r)+\frac{1}{2}\sum_{x,y\in B}\sum_{j=1}^{k}\sum_{l=1}^{k}g_jg_lt_j'(x)t_l'(y)\right\},
\end{aligned}$$

where the last equality follows from (13). Thus,

$$\begin{aligned}
E_N\left[\prod_{x\in B}\prod_{j=1}^{k}\mathrm{e}^{\mathrm{i}t_j'(x)\mathbf{a}^*(x,j)(r)}\right]&=\exp\left\{-\mathrm{i}\sqrt{N}\sum_{x\in B}\sum_{j=1}^{k}\sqrt{a_j(r)}t_j'(x)\right\}\\
&\quad\times\Phi_N\left(\left\{\left(\frac{t_1'(x)}{\sqrt{Na_1(r)}},\ldots,\frac{t_k'(x)}{\sqrt{Na_k(r)}},0,\ldots\right):x\in B\right\}\right)\\
&=(1+o(1))\exp\left\{-\frac{1}{2}\sum_{x\in B}\sum_{j=1}^{k}(t_j'(x))^2+\frac{1}{2}\sum_{x,y\in B}\sum_{j=1}^{k}\sum_{l=1}^{k}g_jg_lt_j'(x)t_l'(y)\right\}\\
&=(1+o(1))\exp\left\{-\frac{1}{2}\sum_{x,y\in B}\sum_{j=1}^{k}\sum_{l=1}^{k}[\delta_{jl}(xy)-g_jg_l]t_j'(x)t_l'(y)\right\}
\end{aligned}$$

for large N. This completes the proof of (i).

In order to prove (ii), let $\alpha=\beta-1$, $b=1-\tilde{r}F(\tilde{r})/\rho$ and

$$\phi=\frac{c}{(\beta-1)(\beta-2)(\beta-3)}.$$

By (8) we can prove that (see the proof of theorem 2 of [11])

$$F(\tilde{r}\,\mathrm{e}^{(\mathrm{i}\theta)})-F(\tilde{r})=\mathrm{i}\tilde{r}F'(\tilde{r})\theta-\phi|\theta|^{\alpha}\exp\left\{-\mathrm{i}\frac{(\alpha-2)\pi}{2}\,\mathrm{sign}(\theta)\right\}+o(|\theta|^{\alpha})$$

and

$$\begin{aligned}
Z_N&=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\exp\left\{\frac{N}{\rho}F(u)-N\log u\right\}u^{-1}\mathrm{d}u\\
&=\frac{\exp\{D_N(\tilde{r})\}}{2\pi}\int_{-\pi}^{\pi}\exp\left\{\frac{N}{\rho}[F(\tilde{r}\,\mathrm{e}^{\mathrm{i}\theta})-F(\tilde{r})]-\mathrm{i}N\theta\right\}\mathrm{d}\theta\\
&=\frac{\exp\{D_N(\tilde{r})\}}{2\pi}\int_{-\pi}^{\pi}\exp\left\{-\mathrm{i}bN\theta-\frac{\phi}{\rho}N|\theta|^{\alpha}\exp\left\{-\mathrm{i}\frac{(\alpha-2)\pi}{2}\,\mathrm{sign}(\theta)\right\}+o(N|\theta|^{\alpha})\right\}\mathrm{d}\theta\\
&=\frac{\exp\{D_N(\tilde{r})\}}{2\pi(\phi N/\rho)^{1/\alpha}}\int_{-\pi(\phi N/\rho)^{1/\alpha}}^{\pi(\phi N/\rho)^{1/\alpha}}\\
&\quad\times\exp\left\{-\mathrm{i}b\frac{N^{(\alpha-1)/\alpha}}{(\phi/\rho)^{1/\alpha}}t+|t|^{\alpha}\exp\left\{-\mathrm{i}\frac{(\alpha-2)\pi}{2}\,\mathrm{sign}(t)\right\}+o(|t|)\right\}\mathrm{d}t
\end{aligned}\tag{21}$$

for $\rho \geqslant \rho_c$ and large N. Note that $1 > 1/\alpha > 1/2$, $N^{1/2} = o(N^{1/\alpha})$ and

$$\frac{N}{|B|\rho}\sum_{x\in B}\sum_{j=1}^{k}[\mathrm{e}^{\mathrm{i}t_j(x)}-1]h(j)\tilde{r}^j\mathrm{e}^{\mathrm{i}(j\theta)} = (1+o(1))\left\{\mathrm{i}\sqrt{N}\sum_{x\in B}\sum_{j=1}^{k}\sqrt{a_j(\tilde{r})}t_j'(x)\right.$$
$$\left.-\frac{1}{2}\sum_{x\in B}\sum_{j=1}^{k}(t_j'(x))^2 + o\left(N^{\frac{1}{\alpha}}\theta\left(\sum_{x\in B}\sum_{j=1}^{k}jt_j'(x)\right)\right)\right\}$$
$$= (1+o(1))\left\{M(\tilde{r}) + o\left(N^{\frac{1}{\alpha}}\theta\left(\sum_{x\in B}\sum_{j=1}^{k}jt_j'(x)\right)\right)\right\} \qquad (22)$$

for large N. It follows from (17)–(20) that

$$\Phi_N\left(\left\{\left(\frac{t_1'(x)}{\sqrt{Na_1(\tilde{r})}},\ldots,\frac{t_k'(x)}{\sqrt{Na_k(\tilde{r})}},0,\ldots\right):x\in B\right\}\right) = (1+o(1))\frac{\exp\{D_N(\tilde{r})+M(\tilde{r})\}}{2\pi Z_N(\phi N/\rho)^{1/\alpha}}$$
$$\times\int_{-\pi(\phi N/\rho)^{1/\alpha}}^{\pi(\phi N/\rho)^{1/\alpha}}\exp\left\{-\mathrm{i}b\frac{N^{(\alpha-1)/\alpha}}{(\phi/\rho)^{1/\alpha}}t + |t|^\alpha\exp\left\{-\mathrm{i}\frac{(\alpha-2)\pi}{2}\,\mathrm{sign}(t)\right\}+o(|t|)\right\}\mathrm{d}t$$
$$= (1+o(1))\exp\{M(\tilde{r})\}.$$

Thus,

$$E_N\left[\prod_{x\in B}\prod_{j=1}^{k}\mathrm{e}^{\mathrm{i}t_j'(x)\mathbf{a}^*(x,j)(\tilde{r})}\right] = \exp\left\{-\mathrm{i}\sqrt{N}\sum_{x\in B}\sum_{j=1}^{k}\sqrt{a_j(\tilde{r})}t_j'(x)\right\}$$
$$\times\Phi_N\left(\left\{\left(\frac{t_1'(x)}{\sqrt{Na_1(\tilde{r})}},\ldots,\frac{t_k'(x)}{\sqrt{Na_k(\tilde{r})}},0,\ldots\right):x\in B\right\}\right)$$
$$= (1+o(1))\exp\left\{-\frac{1}{2}\sum_{x\in B}\sum_{j=1}^{k}(t_j'(x))^2\right\}$$

for $\rho \geqslant \rho_c$ and large N. This is (ii).

By theorem 2 we see that, for any $x, y \in B$ and $j \neq k$, the numbers of the clusters $\mathbf{a}(\mathbf{x},\mathbf{j})$ and $\mathbf{a}(\mathbf{y},\mathbf{k})$ are negative correlative in the sub-critical stage and independent in critical and super-critical stages as $N \to \infty$.

Let a sequence of positive number $\epsilon(N)$ satisfy $\epsilon(N) \to 0$ and $N^{1/2\epsilon(N)}/\log N \to \infty$. Note that $N^{\epsilon(N)} = \log N$ when $\epsilon(N) = \log\log N/\log N$. Let $j_i = d_i N^{\alpha_i}$, where $0 < d_i < d_{i+1} \leqslant \log N$ and $\epsilon(N) \leqslant \alpha_i \leqslant \alpha_{i+1} < 1/\beta$ for $1 \leqslant i \leqslant k$.

By using the same method of proving (ii) of *theorem* 2, we can further obtain theorem 3. Here we omit the proof.

Theorem 3. *Suppose the three conditions in theorem 1 hold. Let* $\mathbf{a}^*(x, j_i) = [\mathbf{a}(\mathbf{x}, j_i) - N^{1-\alpha_i\beta}a_{j_i}]/\sqrt{N^{1-\alpha_i\beta}a_{j_i}}$*, where* $a_{j_i} = c(|B|\rho)^{-1}d_i^{-\beta}, 1 \leqslant i \leqslant k$*. If* $\rho \geqslant \rho_c$*, then* $\{\mathbf{a}^*(x, j_i) : x \in B, 1 \leqslant i \leqslant k\}$ *converges to a mutually independent Gaussian vector* $\{G_i(x) : x \in B, 1 \leqslant i \leqslant k\}$ *as* $N \to \infty$ *with* $E[G_j(x)] = 0$ *and* $E[G_j(x)G_l(y)] = \delta_{jl}(xy)$*.*

Theorem 4. *Suppose the three conditions in theorem 1 hold. Let* $\mathbf{a}(\mathbf{x}, j_l)$*, where* $j_l = b_l N^{1/\beta}$*, where* $0 < b_l < b_{l+1}$ $(1 \leqslant l \leqslant k-1)$*. If* $\rho \geqslant \rho_c$*, then* $\{\mathbf{a}(\mathbf{x}, j_l) : x \in B, 1 \leqslant l \leqslant k\}$ *converges to the mutually independent Poisson vector* $\{P_l(x) : x \in B, 1 \leqslant l \leqslant k\}$ *with the parameter* $c_l(\rho) = E[P_l(x)] = c\big/\big(|B|\rho b_l^\beta\big)$ *for* $x \in B$ *and* $1 \leqslant l \leqslant k$ *as* $N \to \infty$*.*

Proof. Since $1/\alpha > 1/\beta$, we have

$$N^{1/\beta} = o(N^{1/\alpha}), \qquad \mathrm{e}^{\mathrm{i}(j_l\theta)} - 1 = o(N^{1/\alpha}\theta)$$

for large N, where $\alpha = \beta - 1$. Note that $h(j_l)\tilde{r}^{j_l} = (1+o(1))cN^{-1}(b_l)^{-\beta}$. It follows from (17)–(19) that

$$\begin{aligned} E_N\left[\prod_{x\in B}\prod_{l=1}^{k}\mathrm{e}^{\mathrm{i}t_l(x)\mathbf{a}(x,j_l)}\right] &= \Phi_N(\{(t_1(x),\ldots,t_k(x),0,\ldots):x\in B\}) \\ &= \frac{\exp\{D_N(\tilde{r})\}}{2\pi Z_N}\int_{-\pi}^{\pi}\exp\left\{\frac{N}{\rho}[F(\tilde{r}\,\mathrm{e}^{\mathrm{i}\theta}) - F(\tilde{r})] - \mathrm{i}N\theta\right. \\ &\quad \left. + \frac{N}{|B|\rho}\sum_{x\in B}\sum_{l=1}^{k}[\mathrm{e}^{\mathrm{i}t_{j_l}(x)} - 1]h(j_l)\tilde{r}^{j_l}\,\mathrm{e}^{\mathrm{i}(j_l\theta)}\right\}\mathrm{d}\theta \\ &= (1+o(1))\frac{\exp\{D_N(\tilde{r})\}}{2\pi Z_N}\int_{-\pi}^{\pi}\exp\left\{\frac{N}{\rho}[F(\tilde{r}\,\mathrm{e}^{\mathrm{i}\theta}) - F(\tilde{r})] - \mathrm{i}N\theta\right. \\ &\quad \left. + \sum_{x\in B}\sum_{l=1}^{k}c_l(\rho)[\mathrm{e}^{\mathrm{i}t_{j_l}(x)} - 1] + o(N^{1/\alpha}\theta)\right\}\mathrm{d}\theta \\ &= (1+o(1))\exp\left\{\sum_{x\in B}\sum_{l=1}^{k}c_l(\rho)[\mathrm{e}^{\mathrm{i}t_{j_l}(x)} - 1]\right\} \end{aligned}$$

for large N. Thus, the theorem is proved. □

Remark 2. It follows from theorems 2–4 that $\{\mathbf{a}(\mathbf{x}, j_l) : x \in B, 1 \leqslant l \leqslant k\}$ converges to a Gaussian sequence for $j_l = O(N^{1/\delta}), \delta > \beta$, and a Poisson sequence for $j_l = O(N^{1/\beta})$ in the critical and super-critical stages. The clusters with size $j_l = O(N^{1/\delta}), \delta > \beta$, or $j_l = O(N^{1/\delta}), 1 \leqslant \delta \leqslant \beta$, may be respectively called as small or large clusters.

Let $S_N(x) = \sum_{l=1}^{k}\mathbf{a}(\mathbf{x}, j_l)$ and $S_N(x; b_1, b_2) = \sum_{l=k_1}^{k_2}\mathbf{a}(\mathbf{x}, l)$, where $j_l = c_l N^{\nu}, 0 < c_l < c_{l+1}, k_1 = aN^{\nu} - b_1 N^{\beta\nu-1}, k_2 = aN^{\nu} + b_2 N^{\beta\nu-1}, 1/\beta < \nu < 1/(\beta-1)$ and a, b_1, b_2 are three positive constants. Then we have the following results.

Theorem 5. *Suppose the three conditions in theorem 1 hold. If $\rho \geqslant \rho_c$, then*

(i) The probability that $S_N(x) = 0$ for each $x \in B$ converges to 1, that is, $\mu_N(S_N(x) = 0) \to 1$ as $N \to +\infty$;
(ii) $\{S_N(x; b_1, b_2) : x \in B\}$ converges to a mutually independent Poisson vector $\{P(x) : x \in B\}$ with the parameter $E[P(x)] = I_\rho = c[b_2 + b_1](|B|\rho a^{(\beta+1)})^{-1}$ for each $x \in B$ as $N \to +\infty$.

Proof. (i). Let $\Phi_N(\{t(x), x \in B\})$ be the characteristic function of $\{S_N(x), x \in B\}$. Note that $\beta\nu > 1, h(j_l)\tilde{r}^{j_l} = (1+o(1))cN^{-\beta\nu}(c_l)^{-\beta}$ and

$$\exp\{\mathrm{i}(j_l)\theta\} - 1 = \exp\left\{\mathrm{i}\left(\frac{j_l}{N^{1/(\beta-1)}}N^{1/(\beta-1)}\theta\right)\right\} - 1 = o(N^{1/\alpha}\theta)$$

for large N, where $\alpha = \beta - 1$. It follows from (17)–(19) that

$$\Phi_N(\{t(x), x \in B\}) = \frac{\exp\{D_N(\tilde{r})\}}{2\pi Z_N}\int_{-\pi}^{\pi}\exp\left\{\frac{N}{\rho}[F(\tilde{r}\,\mathrm{e}^{\mathrm{i}\theta}) - F(\tilde{r})] - \mathrm{i}N\theta\right.$$

$$+\frac{N}{|B|\rho}\sum_{x\in B}[\mathrm{e}^{\mathrm{i}t(x)}-1]h(j_l)\tilde{r}_l^{j}\,\mathrm{e}^{\mathrm{i}(j_l\theta)}\Bigg\}\,\mathrm{d}\theta,$$

$$=(1+o(1))\exp\left\{N^{-(\beta\nu-1)}\sum_{x\in B}[\mathrm{e}^{\mathrm{i}t(x)}-1]\sum_{l=1}^{k}\frac{c}{|B|\rho(c_l)^{\beta}}\right\}\to 1$$

as $N\to+\infty$.

(ii) Let $\tilde{\Phi}_N(\{t(x),x\in B\})$ be the characteristic function of $\{S_N(x;b_1,b_2),x\in B\}$. As (i) we have

$$\tilde{\Phi}_N(\{t(x),x\in B\})=\frac{\exp\{D_N(\tilde{r})\}}{2\pi Z_N}\int_{-\pi}^{\pi}\exp\Bigg\{\frac{N}{\rho}[F(\tilde{r}\,\mathrm{e}^{\mathrm{i}\theta})-F(\tilde{r})]-\mathrm{i}N\theta$$

$$+\frac{N}{\rho|B|}\sum_{x\in B}[\mathrm{e}^{\mathrm{i}t(x)}-1]\sum_{k=k_1}^{k_2}h(k)\tilde{r}^{k}\,\mathrm{e}^{\mathrm{i}(k\theta)}\Bigg\}\,\mathrm{d}\theta,$$

$$=(1+o(1))\exp\left\{\frac{c}{\rho|B|}N^{1-\nu(\beta-1)}\sum_{x\in B}[\mathrm{e}^{\mathrm{i}t(x)}-1]\int_{a-b_1N^{\nu(\beta-1)-1}}^{a+b_2N^{\nu(\beta-1)-1}}\frac{\mathrm{d}u}{u^{\beta}}\right\}$$

$$\to\ \exp\left\{\sum_{x\in B}I_\rho[\mathrm{e}^{\mathrm{i}t(x)}-1]\right\}$$

as $N\to+\infty$. This completes the proof. □

Theorem 5 shows that though the probability of occurrence of clusters with sizes $j_l=c_lN^{\nu},1\leqslant l\leqslant k$, is zero, the number of clusters with sizes in the large interval $[aN^{\nu}-b_1N^{\beta\nu-1},aN^{\nu}+b_2N^{\beta\nu-1}]$ is subject to the Poisson distribution as $N\to\infty$.

Acknowledgments

We would like to thank the referee for his valuable comments and suggestions that have greatly improved this work. This work was supported by the National Natural Science Foundation of China (10371074).

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